

文章编号:1005-3085(2010)06-1009-06

分数跳-扩散过程下亚式期权定价模型*

薛 红¹, 孙玉东²

(1- 西安工程大学理学院, 西安 710048; 2- 西北工业大学应用数学系, 西安 710129)

摘 要: 本文考虑分数跳-扩散过程下几何平均亚式期权定价问题。首先, 将分数型Itô公式推广到分数跳-扩散情形。其次, 利用分数跳-扩散Itô公式, 给出了分数跳-扩散环境下Black-Scholes偏微分方程。最后, 通过求解偏微分方程, 获得了分数跳-扩散环境下几何平均亚式看涨、看跌期权定价公式。

关键词: 分数跳-扩散过程; 几何平均亚式期权; Black-Scholes偏微分方程

分类号: AMS(2000) 60H10; 90A06

中图分类号: O211.6; F830.9

文献标识码: A

1 引言

金融衍生工具的定价在金融经济学中具有重要地位, 在所有衍生工具中亚式期权交易普遍, 所以亚式期权定价是一个热点问题。亚式期权是一种依赖标的资产价格路径的期权, 它在到期日的收益依赖于期权整个有效期内标的资产的平均价格。文献[1,2]讨论了Brown运动环境下几何平均亚式期权定价模型, 主要采用了偏微分方程的方法, 得到了无风险收益率、红利率、波动率都为常数时几何平均亚式期权定价公式; 文献[3,4]在风险中性测度下采用保险精算方法, 讨论了布朗运动环境下亚式期权定价问题。近年来, 在金融市场中股票价格可能会出现“跳跃”, 不少学者考虑用Poisson过程和布朗运动驱动的随机微分方程来描述股票价格变化并且给出了一些期权价格的解析解^[5]。由于股票价格对过去价格具有依赖性; 一些学者用分数布朗运动刻画股票价格的变化^[6]。本文假定股票价格遵循分数跳-扩散过程, 利用分数跳-扩散过程随机分析理论, 得到了分数-跳扩散环境下几何平均亚式期权价格所满足的Black-Scholes偏微分方程, 并通过求解该偏微分方程, 给出了几何平均亚式看涨、看跌期权的定价公式。

假定在金融市场中有一种债券和一种股票, 债券和股票价格分别满足微分方程

$$dB_t = r_t B_t dt,$$

$$dS_t = S_t((\mu_t - q_t)dt + \sigma(dW_t^H + dN_t)),$$

其中无风险收益率 r_t 、股票期望收益率 μ_t 、股票红利率 q_t 都是时间函数, 波动率 σ 是常数, Q_t 是与 W_t^H 相互独立且强度为 λ 的泊松过程, N_t 是相对应的泊松补偿过程。对自融资投资策略 $\theta = (\theta_t^0, \theta_t^1)$, 财富过程 $V_t = \theta_t^0 B_t + \theta_t^1 S_t$ 满足

$$dV_t = \theta_t^1 dS_t + \theta_t^0 dB_t + \theta_t^1 q_t S_t dt.$$

收稿日期: 2009-01-16. 作者简介: 薛红(1964年8月生), 男, 博士, 教授. 研究方向: 金融数学与集值随机分析.

*基金项目: 陕西省教育厅自然科学专项基金(09JK464).

2 分数跳-扩散过程 Itô 公式

定理 1 假定 $Y_t = W_t^H + N_t$, $f(t, x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R})$, 并且

$$f(t, Y_t), \int_0^t \frac{\partial f}{\partial s}(s, Y_s) ds, \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) ds, \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} ds$$

均属于 $L^2(P)$, 则

$$\begin{aligned} & f(t, Y_t) \\ &= f(0, 0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, Y_s) + H \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} + \frac{\lambda}{2} \frac{\partial^2 f}{\partial x^2}(s, Y_s) \right] ds + \int_0^t \frac{\partial f}{\partial x}(s, Y_s) dW_s. \end{aligned}$$

证明 由于 $Y_t = W_t^H + N_t = W_t^H + Q_t - \lambda t$, 假定 Y 在 $(0, t)$ 时刻内只发生了一次跳, 且跳时刻为 t_1 , 则在 $(0, t_1)$ 和 (t_1, t) 内没有发生跳, 由分数型 Itô 公式^[7] 可得

$$\begin{aligned} & f(t_1, Y_{t_1-}) \\ &= f(0, 0) + \int_0^{t_1} \left[\frac{\partial f}{\partial s}(s, Y_s) - \lambda \frac{\partial f}{\partial x}(s, Y_s) + H \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} \right] ds + \int_0^{t_1} \frac{\partial f}{\partial x}(s, Y_s) dW_s^H, \\ & f(t, Y_t) \\ &= f(t_1, Y_{t_1}) + \int_{t_1}^t \left[\frac{\partial f}{\partial s}(s, Y_s) - \lambda \frac{\partial f}{\partial x}(s, Y_s) + H \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} \right] ds + \int_{t_1}^t \frac{\partial f}{\partial x}(s, Y_s) dW_s^H. \end{aligned}$$

因 $f(t, Y_t)$ 在 t_1 时刻的变化为 $f(t_1, Y_{t_1}) - f(t_1, Y_{t_1-})$, 则有

$$\begin{aligned} f(t, Y_t) &= f(0, 0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, Y_s) - \lambda \frac{\partial f}{\partial x}(s, Y_s) + H \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} \right] ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, Y_s) dW_s^H + f(t_1, Y_{t_1}) - f(t_1, Y_{t_1-}), \end{aligned}$$

从而, 当 $(0, t)$ 内发生跳次数服从泊松过程时, 有

$$\begin{aligned} f(t, Y_t) &= f(0, 0) + \int_0^t \left[\frac{\partial f}{\partial s}(s, Y_s) - \lambda \frac{\partial f}{\partial x}(s, Y_s) + H \frac{\partial^2 f}{\partial x^2}(s, Y_s) s^{2H-1} \right] ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, Y_s) dW_s^H + \sum_{s \leq t} [f(s, Y_s) - f(s, Y_{s-})]. \end{aligned}$$

设 $g(x) \in C^2(\mathbb{R} \rightarrow \mathbb{R})$, 又因为 $\langle dQ_t, dQ_t \rangle = \lambda dt$, 对 $g(Q_t)$ 应用广义 Itô 公式有

$$\sum_{s \leq t} [g(Q_s) - g(Q_{s-})] = \int_0^t g'(Q_t) dQ_t + \frac{\lambda}{2} \int_0^t g''(Q_t) dt,$$

注意到 $Y_t = W_t^H + N_t = W_t^H + Q_t - \lambda t$, 所以

$$\sum_{s \leq t} [f(s, Y_s) - f(s, Y_{s-})] = \int_0^t \frac{\partial f}{\partial x}(s, Y_s) dQ_s + \frac{\lambda}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, Y_s) ds.$$

从而可得结果。

3 分数跳-扩散环境下 Black-Scholes 偏微分方程

定理 2 随机微分方程 $dS_t = S_t((\mu_t - q_t)dt + \sigma(dW_t^H + dN_t))$ 的解为

$$S_t = S_0 \exp \left\{ \int_0^t (\mu_\tau - q_\tau) d\tau - \frac{\lambda \sigma^2}{2} t - \frac{\sigma^2}{2} t^{2H} + \sigma(W_t^H + N_t) \right\}.$$

证明 记

$$f(t, x) = S_0 \exp \left\{ \int_0^t (\mu_\tau - q_\tau) d\tau - \frac{\lambda \sigma^2}{2} t - \frac{\sigma^2}{2} t^{2H} + \sigma x \right\},$$

则 $dS_t = df(t, W_t^H + N_t)$, 由定理 1 可证结果。

定理 3 假定无风险收益率 r_t , 红利率 q_t 是时间 t 的函数, 波动率 σ 是常数, $f(x, y)$ 是损益函数, $F(\cdot, \cdot, \cdot) \in C^{1,2,1}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ 是与损益函数相对应未定权益价值过程, 则分数跳-扩散环境下几何平均亚式期权 Black-Scholes 偏微分方程为

$$\frac{\partial F}{\partial t} + (r_t - q_t)x \frac{\partial F}{\partial x} + y \left(\frac{\ln x - \ln y}{t} \right) \frac{\partial F}{\partial y} + \left(H\sigma^2 t^{2H-1} + \frac{\lambda \sigma^2}{2} \right) x^2 \frac{\partial^2 F}{\partial x^2} - r_t F = 0,$$

$$F(T, x, y) = f(x, y), \quad (t, x, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+.$$

证明 几何平均亚式期权是一个路径依赖期权, 除了与时间 t 以及当时标的资产 S_t 有关以外, 还依赖路径变量

$$J_t = \exp \left\{ \frac{1}{t} \int_0^t \ln S_\tau d\tau \right\},$$

即未定权益的价格过程可表示为 $F = F(t, S_t, J_t)$ 。由于

$$\frac{dJ_t}{dt} = J_t \left(\frac{\ln S_t - \ln J_t}{t} \right),$$

以及定理 1 和定理 2 有

$$\begin{aligned} dF(t, S_t, J_t) = & \left(\frac{\partial F}{\partial t}(t, S_t, J_t) + (\mu_t - q_t)S_t \frac{\partial F}{\partial x}(t, S_t, J_t) + \frac{\ln S_t - \ln J_t}{t} J_t \frac{\partial F}{\partial y}(t, S_t, J_t) \right) dt \\ & + \left(H\sigma^2 t^{2H-1} + \frac{\lambda \sigma^2}{2} \right) S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t, J_t) dt + \sigma S_t \frac{\partial F}{\partial x}(t, S_t, J_t) (dW_t^H + dN_t), \end{aligned}$$

根据自融资交易策略

$$dV_t = \theta_t^0 dB_t + \theta_t^1 dS_t + q_t \theta_t^1 S_t dt = (\theta_t^0 r_t B_t + \theta_t^1 \mu_t S_t) dt + \theta_t^1 \sigma S_t (dW_t^H + dN_t),$$

以及 $V_t = F(t, S_t, J_t)$, 可得偏微分方程

$$\frac{\partial F}{\partial t} + (r_t - q_t)x \frac{\partial F}{\partial x} + \frac{\ln x - \ln y}{t} y \frac{\partial F}{\partial y} + \left(H\sigma^2 t^{2H-1} + \frac{\lambda\sigma^2}{2} \right) x^2 \frac{\partial^2 F}{\partial x^2} - r_t F = 0.$$

由 $F(T, S_T, J_T) = f(S_T, J_T)$ 知, $F(T, x, y) = f(x, y)$, 从而定理得证.

4 模型求解

定理 4 假定无风险收益率 r_t , 红利率 q_t 是时间 t 的函数, 波动率 σ 是常数, 到期日为 T , 执行价格为 K , 则几何平均亚式看涨期权在时刻价格

$$F(t, S_t, J_t) = (J_t^t S_t^{T-t})^{\frac{1}{T}} \exp \left\{ r^*(T-t) - \int_t^T r(\tau) d\tau \right. \\ \left. + \frac{(\sigma_H^*)^2}{2} (T^{2H} - t^{2H}) + \frac{(\sigma_\lambda^*)^2}{2} (T-t) \right\} \Phi(d_1) - K \Phi(d_2),$$

其中

$$d_1 = \frac{\frac{1}{T} \ln \frac{J_t^t S_t^{T-t}}{K^T} + r^*(T-t) + (\sigma_\lambda^*)^2 (T-t) + (\sigma_H^*)^2 (T^{2H} - t^{2H})}{\sqrt{(\sigma_\lambda^*)^2 (T-t) + (\sigma_H^*)^2 (T^{2H} - t^{2H})}},$$

$$d_2 = d_1 - \sqrt{(\sigma_\lambda^*)^2 (T-t) + (\sigma_H^*)^2 (T^{2H} - t^{2H})}, \quad J_t = \exp \left\{ \frac{1}{t} \int_t^T \ln S_\tau d\tau \right\},$$

$$r^* = \frac{\int_t^T (r_\tau - q_\tau) \frac{T-\tau}{T} d\tau}{T-t} - \frac{\lambda\sigma^2 (T-t)}{4T} - \frac{\sigma^2 (T^{2H} - t^{2H})}{2(T-t)} + \frac{H\sigma^2 (T^{2H+1} - t^{2H+1})}{(2H+1)T(T-t)},$$

$$\sigma_H^* = \left[1 - \frac{4H(T^{2H+1} - t^{2H+1})}{(2H+1)T(T^{2H} - t^{2H})} + \frac{H(T^{2H+2} - t^{2H+2})}{(H+1)T^2(T^{2H} - t^{2H})} \right]^{\frac{1}{2}} \sigma, \quad \sigma_\lambda^* = \frac{(T-t)}{\sqrt{3}T} \sqrt{\lambda} \sigma.$$

证明 取 $f(x, y) = (y - K)^+$, 记 $U(t, \xi) = F(t, x, y)$, 其中

$$\xi = \frac{t \ln y + (T-t) \ln x}{T},$$

则分数跳-扩散环境下 Black-Scholes 偏微分方程化为

$$\frac{\partial U}{\partial t} + \left(H\sigma^2 t^{2H-1} + \frac{\lambda\sigma^2}{2} \right) \left(\frac{T-t}{T} \right)^2 \frac{\partial^2 U}{\partial \xi^2} \\ + \left(r_t - q_t - H\sigma^2 t^{2H-1} - \frac{\lambda\sigma^2}{2} \right) \left(\frac{T-t}{T} \right) \frac{\partial U}{\partial \xi} - r_t U = 0,$$

$$U(T, \xi) = (e^\xi - K)^+.$$

令

$$\begin{aligned}
 z &= \xi + \int_t^T (r_\tau - q_\tau) \frac{T-\tau}{T} d\tau - \frac{\lambda \sigma^2}{4T} (T-t)^2 - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) \\
 &\quad + \frac{H\sigma^2}{T(2H+1)} (T^{2H+1} - t^{2H+1}), \\
 s &= \frac{\lambda \sigma^2}{6T^2} (T-t)^3 - \frac{\sigma^2}{2} (T^{2H} - t^{2H}) + \frac{2H\sigma^2}{T(2H+1)} (T^{2H+1} - t^{2H+1}) \\
 &\quad - \frac{H\sigma^2}{2(H+1)T^2} (T^{2H+2} - t^{2H+2}), \\
 C(s, z) &= U(t, \xi) \exp \left\{ \int_t^T r_\tau d\tau \right\},
 \end{aligned}$$

则分数跳-扩散环境下 Black-Scholes 偏微分方程可化为

$$\frac{\partial C}{\partial s} = \frac{\partial^2 C}{\partial z^2}, \quad C(0, z) = (e^z - K)^+.$$

根据热传导方程经典解理论可得

$$C(s, z) = e^{z+s} \Phi\left(\frac{z+2s-\ln K}{\sqrt{2s}}\right) - K \Phi\left(\frac{z-\ln K}{\sqrt{2s}}\right),$$

对上面的变换进行逆变换可得定理结果。

注1 当 $H = \frac{1}{2}$ 时, 可得跳-扩散过程下几何平均亚式看涨期权定价公式。特别地, 当 q, r 为常数, $\lambda = 0$ 时, 可得文献 [2] 中结果。

定理5 假定无风险收益率 r_t , 红利率 q_t 是时间 t 的函数, 波动率 σ 是常数, 到期日为 T , 执行价格为 K , 则几何平均亚式看跌期权定价公式为

$$\begin{aligned}
 C(t, S_t, J_t) &= -(J_t^t S_t^{T-t})^{\frac{1}{2}} \exp \left\{ r^*(T-t) - \int_t^T r(\tau) d\tau \right. \\
 &\quad \left. + \frac{(\sigma_H^*)^2}{2} (T^{2H} - t^{2H}) + \frac{(\sigma_\lambda^*)^2}{2} (T-t) \right\} N(-d_1) + KN(-d_2),
 \end{aligned}$$

其中 $d_1, d_2, J_t, r^*, \sigma_H^*, \sigma_\lambda^*$ 见定理4。

证明 取 $f(x, y) = (K - y)^+$, 按照定理4思路可证。

注2 当 $H = \frac{1}{2}$ 时, 可得跳-扩散过程下几何平均亚式看跌期权定价公式。特别地, 当 q, r 为常数, $\lambda = 0$ 时, 可得文献 [2] 中结果。

参考文献:

- [1] 王志明, 朱芳芳. 几何平均亚式期权定价模型的新解法[J]. 武汉科技大学学报(自然科学版), 2008, 32(2): 321-322
Wang Z M, Zhu F F. A new solution to the geometric average Asian option pricing model[J]. Journal of Wuhan University of Science and Technology (Natural Science Edition), 2008, 32(2): 321-322

- [2] 姜礼尚. 期权定价的数学模型和方法(第二版)[M]. 北京: 高等教育出版社, 2008
Jiang L S. Mathematical Modeling and Methods of Option Pricing (Second Edition)[M]. Beijing: Higher Education Press, 2008
- [3] 罗庆红, 杨向群. 几何型亚式期权定价研究[J]. 湖南文理学院学报, 2007, 19(1): 5-7
Luo Q H, Yang X Q. The study on Asian geometric average options[J]. Journal of Hunan University of Arts and Science, 2007, 19(1): 5-7
- [4] 王国强, 马德全, 宋华. 亚式期权的一种定价方法[J]. 数学理论与应用, 2007, 27(3): 410-411
Wang G Q, Ma D Q, Song H. One method for Asian option pricing[J]. Mathematical Theory and Applications, 2007, 27(3): 410-411
- [5] 陈超. 标的资产价格服从跳-扩散过程的脆弱期权定价模型[J]. 工程数学学报, 2008, 25(6): 1129-1132
Chen C. Vulnerable european option pricing models when underlying asset returns are jump-diffusion processes[J]. Chinese Journal of Engineering Mathematics, 2008, 25(6): 1129-1132
- [6] 薛红, 王拉省. 分数布朗运动环境中最值期权定价[J]. 工程数学学报, 2008, 25(5): 843-850
Xue H, Wang L S. Pricing of maximum or minimum option in the fractional brownian motion environment[J]. Chinese Journal of Engineering Mathematics, 2008, 25(5): 843-849
- [7] Biagini F, Hu Y, et al. Stochastic Calculus for Fractional Brownian Motion and Applications[M]. New York: Springer, 2008

Asian Option Pricing Model in Fractional Jump-diffusion Environment

XUE Hong¹, SUN Yu-dong²

(1- School of Science, Xi'an Polytechnic University, Xi'an 710048;

2- Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710129)

Abstract: This paper considers the pricing problem of the geometric average Asian option. First, the fractional Itô formula is generalized to the fractional jump-diffusion processes case. Then, the Black-Scholes partial differential equation in the fractional jump-diffusion environment is obtained by Itô formula for fractional jump-diffusion processes. Finally, the pricing formulae of the geometric average Asian call and put options are obtained by the partial differential equation theory.

Keywords: fractional jump-diffusion process; geometric average Asian option; Black-Scholes partial differential equation

Received: 16 Jan 2009. **Accepted:** 21 Apr 2010.

Foundation item: The Program of Natural Science Foundation of the Education Department of Shaanxi Province, China (09JK464).